GROUP REPRESENTATIONS

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Representation theory is the study of the various ways a given group can be mapped into a general linear group. This information has proven to be effective at providing insight into the structure of the given group as well as the objects on which the group acts. Most notable is the central contribution made by representation theory to the complete classification of finite simple groups [Gor94]. (See also Fact 3 of subsection 1 and Fact 5 of subsection 6.) Representations of finite groups can be defined over an arbitrary field and such have been studied extensively. Here, however, we discuss only the most widely used, classical theory of representations over the field of complex numbers (many results of which fail to hold over other fields).

1 BASIC CONCEPTS

Throughout, G denotes a finite group, e denotes its identity element, and V denotes a finite dimensional complex vector space.

Definitions:

The **general linear group** of a vector space V is the group $GL(V)$ of linear isomorphisms of V onto itself with operation given by function composition.
A (**linear**) **representation** of the finite group G (over the complex field $\mathbb{C}$) is a homomorphism $\rho = \rho_V : G \to GL(V)$, where V is a finite dimensional vector space over $\mathbb{C}$.
The **degree** of a representation ρ_V is the dimension of the vector space V.
Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are **equivalent** (or **isomorphic**) if there exists a linear isomorphism $\tau : V \to V'$ such that $\tau \circ \rho(s) = \rho'(s) \circ \tau$ for all $s \in G$.
Given a representation ρ_V of G, a subspace W of V is G-**stable** (or G-**invariant**) if $\rho_V(s)(W) \subseteq W$ for all $s \in G$.

If ρ_V is a representation of G and W is a G-stable subspace of V, then the induced maps $\rho_W : G \to GL(W)$ and $\rho_{V/W} : G \to GL(V/W)$ are the corresponding **subrepresentation** and **quotient representation**, respectively.
A representation ρ_V of G with $V \neq \{0\}$ is **irreducible** if V and $\{0\}$ are the only G-stable subspaces of V; otherwise, ρ_V is **reducible**.
The **kernel** of a representation ρ_V of G is the set of all $s \in G$ for which $\rho_V(s) = 1_V$.
A representation of G is **faithful** if its kernel consists of the identity element alone.
An **action** of G on a set X is a function $G \times X \to X$, $(s, x) \mapsto sx$, satisfying

- $(st)x = s(tx)$ for all $s, t \in G$ and $x \in X$,
- $ex = x$ for all $x \in X$.

A $\mathbb{C}G$**-module** is a finite-dimensional vector space V over $\mathbb{C}$ together with an action $(s, v) \mapsto sv$ of G on V that is linear in the variable v, meaning

- $s(v + w) = sv + sw$ for all $s \in G$ and $v, w \in V$,
- $s(\alpha v) = \alpha(sv)$ for all $s \in G$, $v \in V$ and $\alpha \in \mathbb{C}$.

(See Fact 6.)

Facts:

The following facts can be found in [Isa94, pp. 4 - 10] or [Ser77, pp. 3 - 13, 47].

1. If $\rho = \rho_V$ is a representation of G, then

- $\rho(e) = 1_V$,
- $\rho(st) = \rho(s)\rho(t)$ for all $s, t \in G$,
- $\rho(s^{-1}) = \rho(s)^{-1}$ for all $s \in G$.

2. A representation of G of degree one is a group homomorphism from G into the group $\mathbb{C}^\times$ of nonzero complex numbers under multiplication (identifying $\mathbb{C}^\times$ with $GL(\mathbb{C})$). Every representation of degree one is irreducible.

3. The group G is abelian if and only if every irreducible representation of G is of degree one.

4. *Maschke's Theorem*: If ρ_V is a representation of G and W is a G-stable subspace of V, then there exists a G-stable vector space complement of W in V.

5. *Schur's Lemma*: Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be two irreducible representations of G and let $f : V \to V'$ be a linear map satisfying $f \circ \rho(s) = \rho'(s) \circ f$ for all $s \in G$.

- If ρ' is not equivalent to ρ, then f is the zero map.
- If $V' = V$ and $\rho' = \rho$, then f is a scalar multiple of the identity map: $f = \alpha 1_V$ for some $\alpha \in \mathbb{C}$.

6. If $\rho = \rho_V$ is a representation of G, then V becomes a $\mathbb{C}G$-module with action given by $sv = \rho(s)(v)$ ($s \in G$, $v \in V$). Conversely, if V is a $\mathbb{C}G$-module, then $\rho_V(s)(v) = sv$ defines a representation $\rho_V : G \to GL(V)$ (called the **representation of G afforded by V**). The study of representations of the finite group G is the same as the study of $\mathbb{C}G$-modules.

7. The vector space $\mathbb{C}G$ over $\mathbb{C}$ with basis G is a ring (the **group ring** of G over $\mathbb{C}$) with multiplication obtained by linearly extending the operation in G to arbitrary products. If V is a $\mathbb{C}G$-module and the action of G on V is extended linearly to a map $\mathbb{C}G \times V \to V$, then V becomes a (left unitary) $\mathbb{C}G$-module in the ring theoretic sense, that is, V satisfies the usual vector space axioms (see article cite(1.1 Vectors, Matrices and Systems of Linear Equations)) with the scalar field replaced by the ring $\mathbb{C}G$.

Examples:

See also examples in the next subsection.

1. Let $n \in \mathbf{N}$ and let $\omega \in \mathbb{C}$ be an nth root of unity (meaning $\omega^n = 1$). Then the map $\rho : \mathbf{Z}_n \to \mathbb{C}^\times$ given by $\rho(m) = \omega^m$ is a representation of degree one of the group $\mathbf{Z}_n$ of integers modulo n. It is irreducible.

2. *Regular representation*: Let $V = \mathbb{C}G$ be the complex vector space with basis G. For each $s \in G$ there is a unique linear map $\rho(s) : V \to V$ satisfying $\rho(s)(t) = st$ for all $t \in G$. Then $\rho : G \to GL(V)$ is a representation of G called the (**left**) **regular representation**. If $|G| > 1$, then the regular representation is reducible (see Example 3 of subsection 5) .

3. *Permutation representation*: Let X be a finite set, let $(s, x) \mapsto sx$ be an action of G on X, and let V be the complex vector space with basis X. For each $s \in G$ there is a unique linear map $\rho(s) : V \to V$ satisfying $\rho(s)(x) = sx$ for all $x \in X$. Then $\rho : G \to GL(V)$ is a representation of G called a **permutation representation**. The regular representation of G (Example 2) is the permutation representation corresponding to the action of G on itself given by left multiplication.

4. The representation of G of degree 1 given by $\rho(s) = 1 \in \mathbb{C}^\times$ for all $s \in G$ is the **trivial representation**.

5. *Direct sum*: If V and W are $\mathbb{C}G$-modules, then the $\mathbb{C}$-vector space direct sum $V \oplus W$ is a $\mathbb{C}G$-module with action given by $s(v, w) = (sv, sw)$ ($s \in G$, $v \in V$, $w \in W$).

6. *Tensor product*: If V_1 is a $\mathbb{C}G_1$-module and V_2 is a $\mathbb{C}G_2$-module, then the $\mathbb{C}$-vector space tensor product $V_1 \otimes V_2$ is a $\mathbb{C}(G_1 \times G_2)$-module with action given by $(s_1, s_2)(v_1 \otimes v_2) = (s_1v_1) \otimes (s_2v_2)$ ($s_i \in G_i$, $v_i \in V_i$). If both groups G_1 and G_2 equal the same group G, then $V_1 \otimes V_2$ is a $\mathbb{C}G$-module with action given by $s(v_1 \otimes v_2) = (sv_1) \otimes (sv_2)$ ($s \in G$, $v_i \in V_i$).

7. *Contragredient*: If V is a $\mathbb{C}G$-module, then the $\mathbb{C}$-vector space dual V^* is a $\mathbb{C}G$-module (called the **contragredient** of V) with action given by $(sf)(v) = f(s^{-1}v)$ ($s \in G$, $f \in V^*$, $v \in V$).

2 MATRIX REPRESENTATIONS

Throughout, G denotes a finite group, e denotes its identity element, and V

denotes a finite dimensional complex vector space.

Definitions:

A **matrix representation** of G of **degree** n (over the field $\mathbb{C}$) is a homomorphism $R : G \to GL_n(\mathbb{C})$, where $GL_n(\mathbb{C})$ is the group of nonsingular $n \times n$ matrices over the field $\mathbb{C}$. (If $n = 0$, then $GL_n(\mathbb{C})$ is interpreted as the trivial group $\{\emptyset\}$.) (For the relationship between representations and matrix representations, see the facts below.)
Two matrix representations R and R' are **equivalent** (or **isomorphic**) if they have the same degree, say n, and there exists a nonsingular $n \times n$ matrix P such that $R'(s) = PR(s)P^{-1}$ for all $s \in G$.
A matrix representation of G is **reducible** if it is equivalent to a matrix representation R having the property that for each $s \in G$, the matrix $R(s)$ has the block form

$$R(s) = \begin{bmatrix} X(s) & Z(s) \\ 0 & Y(s) \end{bmatrix}$$

(block sizes independent of s).
A matrix representation is **irreducible** if it has nonzero degree and it is not reducible.
The **kernel** of a matrix representation R of G of degree n is the set of all $s \in G$ for which $R(s) = I_n$.
A matrix representation of G is **faithful** if its kernel consists of the identity element alone.

Facts:

The following facts can be found in [Isa94, pp. 10 - 11, 32] or [Ser77, pp. 11 - 14].

1. If R is a matrix representation of G, then

- $R(e) = I$,
- $R(st) = R(s)R(t)$ for all $s, t \in G$,

- $R(s^{-1}) = R(s)^{-1}$ for all $s \in G$.

2. If $\rho = \rho_V$ is a representation of G of degree n and $\mathcal{B}$ is an ordered basis for V, then $R_{\rho,\mathcal{B}}(s) = [\rho(s)]_{\mathcal{B}}$ defines a matrix representation $R_{\rho,\mathcal{B}} : G \to GL_n(\mathbb{C})$ called the **matrix representation of G afforded by the representation** ρ (or **by the $\mathbb{C}G$-module V**) **with respect to the basis $\mathcal{B}$**. Conversely, if R is a matrix representation of G of degree n and $V = \mathbb{C}^n$, then $\rho(s)(v) = R(s)v$ $(s \in G,\, v \in V)$ defines a representation ρ of G and $R = R_{\rho,\mathcal{B}}$, where $\mathcal{B}$ is the standard ordered basis of V.

3. If R and R' are matrix representations afforded by representations ρ and ρ', respectively, then R and R' are equivalent if and only if ρ and ρ' are equivalent. In particular, two matrix representations that are afforded by the same representation are equivalent regardless of the chosen bases.

4. If $\rho = \rho_V$ is a representation of G and W is a G-stable subspace of V and a basis for W is extended to a basis $\mathcal{B}$ of V, then for each $s \in G$ the matrix $R_{\rho,\mathcal{B}}(s)$ is of block form

$$R_{\rho,\mathcal{B}}(s) = \begin{bmatrix} X(s) & Z(s) \\ 0 & Y(s) \end{bmatrix},$$

where X and Y are the matrix representations afforded by ρ_W (with respect to the given basis) and $\rho_{V/W}$ (with respect to the induced basis), respectively.

5. If the matrix representation R of G is afforded by a representation ρ, then R is irreducible if and only if ρ is irreducible.

6. The group G is abelian if and only if every irreducible matrix representation of G is of degree one .

7. *Maschke's Theorem* (*for matrix representations*): If R is a matrix representation of G and for each $s \in G$ the matrix $R(s)$ is of block form

$$R(s) = \begin{bmatrix} X(s) & Z(s) \\ 0 & Y(s) \end{bmatrix}$$

(block sizes independent of s), then R is equivalent to the matrix representation R' given by

$$R'(s) = \begin{bmatrix} X(s) & 0 \\ 0 & Y(s) \end{bmatrix}$$

$(s \in G)$.

8. *Schur relations*: Let R and R' be irreducible matrix representations of G of degrees n and n', respectively. For $1 \leq i, j \leq n$ and $1 \leq i', j' \leq n'$ define functions $r_{ij}, r'_{i'j'} : G \to \mathbb{C}$ by $R(s) = [r_{ij}(s)]$, $R'(s) = [r'_{i'j'}(s)]$ $(s \in G)$.

- If R' is not equivalent to R, then for all $1 \leq i, j \leq n$ and $1 \leq i', j' \leq n'$

$$\sum_{s \in G} r_{ij}(s^{-1}) r'_{i'j'}(s) = 0.$$

- For all $1 \leq i, j, k, l \leq n$

$$\sum_{s \in G} r_{ij}(s^{-1}) r_{kl}(s) = \begin{cases} |G|/n & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Examples:

1. An example of a degree two matrix representation of the symmetric group S_3 is given by

$$R(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad R((23)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$R((13)) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad R((123)) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad R((132)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

(cf. Example 2 of subsection 4.)

2. The matrix representation R of the additive group $\mathbf{Z}_3$ of integers modulo 3 afforded by the regular representation with respect to the basis $\mathbf{Z}_3 = \{0, 1, 2\}$ (ordered as indicated) is given by

$$R(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R(1) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad R(2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

3. Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be two representations of G, let $\mathcal{B}$ and $\mathcal{B}'$ be bases of V and V', respectively, and let $R = R_{\rho,\mathcal{B}}$ and $R' = R_{\rho',\mathcal{B}'}$ be the afforded matrix representations.

- The matrix representation afforded by the direct sum $V \oplus V'$ with respect to the basis $\{(b, 0), (0, b') \mid b \in \mathcal{B}, b' \in \mathcal{B}'\}$ is given by $s \mapsto R(s) \oplus R'(s)$ (direct sum of matrices).
- The matrix representation afforded by the tensor product $V \otimes V'$ with respect to the basis $\{b \otimes b' \mid b \in \mathcal{B}, b' \in \mathcal{B}'\}$ is given by $s \mapsto R(s) \otimes R'(s)$ (Kronecker product of matrices).
- The matrix representation afforded by the contragredient V^* with respect to the dual basis of $\mathcal{B}$ is given by $s \mapsto (R(s)^{-1})^T$ (inverse transpose of matrix).

3 CHARACTERS

Throughout, G denotes a finite group, e denotes its identity element, and V denotes a finite dimensional complex vector space.

Definitions:

The **character of** G **afforded by a matrix representation** R of G is the function $\chi : G \to \mathbb{C}$ defined by $\chi(s) = \text{tr } R(s)$.
The **character of** G **afforded by a representation** $\rho = \rho_V$ of G is the character afforded by the corresponding matrix representation $R_{\rho,\mathcal{B}}$, where $\mathcal{B}$ is a basis for V.
The **character of** G **afforded by a** $\mathbb{C}G$**-module** V is the character afforded by the corresponding representation ρ_V.
An **irreducible character** is a character afforded by an irreducible representation.
The **degree** of a character χ of G is the number $\chi(e)$.
A **linear character** is a character of degree one.

If χ_1 and χ_2 are two characters of G, their **sum** is defined by $(\chi_1+\chi_2)(s) = \chi_1(s)+\chi_2(s)$ and their **product** is defined by $(\chi_1\chi_2)(s) = \chi_1(s)\chi_2(s)$ $(s \in G)$.
If χ is a character of G, its **complex conjugate** is defined by $\overline{\chi}(s) = \overline{\chi(s)}$ $(s \in G)$, where $\overline{\chi(s)}$ denotes the conjugate of the complex number $\chi(s)$.
The **kernel** of a character χ of G is the set $\{s \in G \,|\, \chi(s) = \chi(e)\}$.
A character χ of G is **faithful** if its kernel consists of the identity element alone.
The **principal character** of G is the character 1_G satisfying $1_G(s) = 1$ for all $s \in G$.
If χ and ψ are two characters of G, then ψ is called a **constituent** of χ if $\chi = \psi + \psi'$ with ψ' a character (possibly zero) of G.

Facts:

The following facts can be found in [Isa94, pp. 14 - 23, 38 - 40, 59] or [Ser77, pp. 10 - 19, 27, 52].

1. The degree of a character of G equals the dimension of V, where ρ_V is a representation affording the character.

2. If χ is a character of G, then $\chi(s^{-1}) = \overline{\chi(s)}$ and $\chi(t^{-1}st) = \chi(s)$ for all $s, t \in G$.

3. Two characters of G are equal if and only if representations affording them are equivalent.

4. The number of distinct irreducible characters of G is the same as the number of conjugacy classes of G.

5. Every character χ of G can be expressed in the form $\chi = \sum_{\varphi \in Irr(G)} m_\varphi \varphi$, where $Irr(G)$ denotes the set of irreducible characters of G and where each m_φ is a nonnegative integer (called the **multiplicity** of φ as a constituent of χ).

6. A nonzero character of G is irreducible if and only if it is not the sum of two nonzero characters of G.

7. The kernel of a character equals the kernel of a representation affording the character.

8. The degree of an irreducible character of G divides the order of G.

9. A character of G is linear if and only if it is a homomorphism from G into the multiplicative group of nonzero complex numbers under multiplication.

10. The group G is abelian if and only if every irreducible character of G is linear.

11. The sum of the squares of the irreducible character degrees equals the order of G.

12. *Irreducible characters of direct products*: Let G_1 and G_2 be finite groups. Denoting by $Irr(G)$ the set of irreducible characters of the group G, we have $Irr(G_1 \times G_2) = Irr(G_1) \times Irr(G_2)$, where an element (χ_1, χ_2) of the Cartesian product on the right is viewed as a function on the direct product $G_1 \times G_2$ via $(\chi_1, \chi_2)(s_1, s_2) = \chi_1(s_1)\chi_2(s_2)$.

13. *Burnside's Vanishing Theorem*: If χ is a nonlinear irreducible character of G, then $\chi(s) = 0$ for some $s \in G$.

Examples:

See also examples in the next subsection.

1. If V_1 and V_2 are $\mathbb{C}G$-modules and χ_1 and χ_2, respectively, are the characters of G they afford, then the direct sum $V_1 \oplus V_2$ affords the sum $\chi_1 + \chi_2$ and the tensor product $V_1 \otimes V_2$ affords the product $\chi_1\chi_2$.

2. If V is a $\mathbb{C}G$-module and χ is the character it affords, then the contragredient V^* affords the complex conjugate character $\overline{\chi}$.

3. Let X be a finite set on which an action of G is given and let ρ be the corresponding permutation representation of G (see Example 3 of subsection 1). If χ is the character afforded by ρ, then for each $s \in G$, $\chi(s)$ is the number of ones on the main diagonal of the permutation matrix $[\rho(s)]_X$, which is the same as the number of fixed points of X under the action of s: $\chi(s) = |\{x \in X \mid sx = x\}|$. The matrix representation of $\mathbf{Z}_3$ given in Example 2 of subsection 2 is afforded by a permutation representation, namely, the regular representation; it affords the character χ given by $\chi(0) = 3$, $\chi(1) = 0$, $\chi(2) = 0$ in accordance with the statement above.

4 ORTHOGONALITY RELATIONS AND CHARACTER TABLE

Throughout, G denotes a finite group, e denotes its identity element, and V denotes a finite dimensional complex vector space.

Definitions:

A function $f : G \to \mathbb{C}$ is called a **class function** if it is constant on the conjugacy classes of G, that is, if $f(t^{-1}st) = f(s)$ for all $s, t \in G$.
The **inner product** of two functions f and g from G to $\mathbb{C}$ is the complex number

$$(f, g)_G = \frac{1}{|G|} \sum_{s \in G} f(s)\overline{g(s)}.$$

The **character table** of the group G is the square array with entry in the ith row and jth column equal to the complex number $\chi_i(c_j)$, where $Irr(G) = \{\chi_1, \ldots, \chi_k\}$ is the set of distinct irreducible characters of G and $\{c_1, \ldots, c_k\}$ is a set consisting of exactly one element from each conjugacy class of G.

Facts:

The following facts can be found in [Isa94, pp. 14 - 21, 30] or [Ser77, pp.10 - 19].

1. Each character of G is a class function.

2. *First Orthogonality Relation*: If φ and ψ are two irreducible characters of G, then

$$(\phi, \psi)_G = \frac{1}{|G|} \sum_{s \in G} \varphi(s)\overline{\psi(s)} = \begin{cases} 1 & \text{if } \varphi = \psi \\ 0 & \text{if } \varphi \neq \psi. \end{cases}$$

3. *Second Orthogonality Relation*: If s and t are two elements of G, then

$$\sum_{\chi \in Irr(G)} \chi(s)\overline{\chi(t)} = \begin{cases} |G|/c(s) & \text{if } t \text{ is conjugate to } s \\ 0 & \text{if } t \text{ is not conjugate to } s, \end{cases}$$

where $c(s)$ denotes the number of elements in the conjugacy class of s.

4. *Generalized Orthogonality Relation*: If φ and ψ are two irreducible characters of G and t is an element of G, then

$$\frac{1}{|G|}\sum_{s\in G}\varphi(st)\overline{\psi(s)} = \begin{cases} \varphi(t)/\varphi(e) & \text{if } \varphi=\psi \\ 0 & \text{if } \varphi\neq\psi. \end{cases}$$

This generalizes the First Orthogonality Relation (Fact 2).

5. The set of complex-valued functions on G is a complex inner product space with inner product as defined above. The set of class functions on G is a subspace.

6. A character χ of G is irreducible if and only if $(\chi,\chi)_G = 1$.

7. The set $Irr(G)$ of irreducible characters of G is an orthonormal basis for the inner product space of class functions on G.

8. If the character χ of G is expressed as a sum of irreducible characters (see Fact 5 of subsection 3), then the number of times the irreducible character φ appears as a summand is $(\chi,\varphi)_G$. In particular, $\varphi \in Irr(G)$ is a constituent of χ if and only if $(\chi,\varphi)_G \neq 0$.

9. Isomorphic groups have identical character tables (up to a reordering of rows and columns). The converse of this statement does not hold since, for example, the dihedral group and the quaternion group (both of order eight) have the same character table, yet they are not isomorphic.

Examples:

1. The character table of the group $\mathbf{Z}_4$ of integers modulo four is

	0	1	2	3
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

2. The character table of the symmetric group S_3 is

	(1)	(12)	(123)
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

Note that χ_2 is the character afforded by the matrix representation of S_3 given in Example 1 of subsection 2.

3. The character table of the symmetric group S_4 is

	(1)	(12)	(12)(34)	(123)	(1234)
χ_0	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	2	0	2	-1	0
χ_3	3	1	-1	0	-1
χ_4	3	-1	-1	0	1

4. The character table of the alternating group A_4 is

	(1)	(12)(34)	(123)	(132)
χ_0	1	1	1	1
χ_1	1	1	ω	ω^2
χ_2	1	1	ω^2	ω
χ_3	3	-1	0	0

where $\omega = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

5. Let ρ_V be a representation of G and for each irreducible character φ of G put

$$T_\varphi = \frac{\varphi(e)}{|G|} \sum_{s \in G} \varphi(s^{-1})\rho_V(s) : V \to V.$$

Then the Generalized Orthogonality Relation (Fact 4) shows that

$$T_\varphi T_\psi = \begin{cases} T_\varphi & \text{if } \varphi = \psi \\ 0 & \text{if } \varphi \neq \psi \end{cases}$$

and

$$\sum_{\varphi \in Irr(G)} T_\varphi = 1_V,$$

where 1_V denotes the identity operator on V. Moreover, $V = \bigoplus_{\varphi \in Irr(G)} T_\varphi(V)$ (internal direct sum).

5 RESTRICTION AND INDUCTION OF CHARACTERS

Throughout, G denotes a finite group, e denotes its identity element, and V denotes a finite dimensional complex vector space.

Definitions:

If χ is a character of G and H is a subgroup of G, then the **restriction** of χ to H is the character χ_H of H obtained by restricting the domain of χ.
A character φ of a subgroup H of G is **extendible** to G if $\varphi = \chi_H$ for some character χ of G.
If φ is a character of a subgroup H of G, then the **induced character** from H to G is the character φ^G of G given by the formula

$$\varphi^G(s) = \frac{1}{|H|} \sum_{t \in G} \varphi^\circ(t^{-1}st),$$

where φ° is defined by $\varphi^\circ(x) = \varphi(x)$ if $x \in H$ and $\varphi^\circ(x) = 0$ if $x \notin H$.
If φ is a character of a subgroup H of G and s is an element of G, then the **conjugate character** of φ by s is the character φ^s of $H^s = s^{-1}Hs$ given by $\varphi^s(h^s) = \varphi(h)$ $(h \in H)$, where $h^s = s^{-1}hs$.

Facts:

The following facts can be found in [Isa94, pp. 62 - 63, 73 - 79] or [Ser77, pp. 55 - 58].

1. The restricted character defined above is indeed a character: χ_H is afforded by the restriction to H of a representation affording χ.

2. The induced character defined above is indeed a character: Let V be a $\mathbb{C}H$-module affording φ and put $V^G = \bigoplus_{t \in T} V_t$, where $G = \bigcup_{t \in T} tH$ (disjoint union) and $V_t = V$ for each t. Then V^G is a $\mathbb{C}G$-module that affords φ^G, where the action is given as follows: for $s \in G$ and $v \in V_t$, sv is the element hv of $V_{t'}$, where $st = t'h$ ($t' \in T$, $h \in H$).

3. The conjugate character defined above is indeed a character: φ^s is afforded by the representation of H^s obtained by composing the homomorphism $H^s \to H$, $h^s \mapsto h$ with a representation affording φ.

4. *Additivity of restriction*: Let H be a subgroup of G. If χ and χ' are characters of G, then $(\chi + \chi')_H = \chi_H + \chi'_H$.

5. *Additivity of induction*: Let H be a subgroup of G. If φ and φ' are characters of H, then $(\varphi + \varphi')^G = \varphi^G + \varphi'^G$.

6. *Transitivity of induction*: Let H and K be subgroups of G with $H \subseteq K$. If φ is a character of H, then $(\varphi^K)^G = \varphi^G$.

7. *Degree of induced character*: If H is a subgroup of G and φ is a character of H, then the degree of the induced character φ^G equals the product of the index of H in G and the degree of φ: $\varphi^G(e) = [G : H]\varphi(e)$.

8. Let χ be a character of G and let H be a subgroup of G. If the restriction χ_H is irreducible, then so is χ. The converse of this statement does not hold. In fact, if H is the trivial subgroup then $\chi_H = \chi(e)1_H$, so any nonlinear irreducible character (e.g., χ_2 in Example 2 of subsection 4) provides a counterexample.

9. Let H be a subgroup of G and let φ be a character of H. If the induced character φ^G is irreducible, then so is φ. The converse of this statement does not hold (see Example 3).

10. Let H be a subgroup of G. If φ is an irreducible character of H, then there exists an irreducible character χ of G such that φ is a constituent of χ_H.

11. *Frobenius Reciprocity*: If χ is a character of G and φ is a character of a subgroup H of G, then $(\varphi^G, \chi)_G = (\varphi, \chi_H)_H$.

12. If χ is a character of G and φ is a character of a subgroup H of G, then $(\varphi\chi_H)^G = \varphi^G\chi$.

13. *Mackey's Subgroup Theorem*: If H and K are subgroups of G and φ is a character of H, then $(\varphi^G)_K = \sum_{t\in T}(\varphi^t_{H^t\cap K})^K$, where T is a set of representatives for the (H, K)-double cosets in G (so that $G = \dot{\bigcup}_{t\in T} HtK$, a disjoint union).

14. If φ is a character of a normal subgroup N of G, then for each $s \in G$, the conjugate φ^s is a character of N. Moreover, $\varphi^s(n) = \varphi(sns^{-1})$ $(n \in N)$.

15. *Clifford's Theorem*: Let N be a normal subgroup of G, let χ be an irreducible character of G, and let φ be an irreducible constituent of χ_N. Then $\chi_N = m\sum_{i=1}^{h}\varphi_i$, where $\varphi_1, \ldots, \varphi_h$ are the distinct conjugates of φ under the action of G and $m = (\chi_N, \varphi)_N$.

Examples:

1. Given a subgroup H of G, the induced character $(1_H)^G$ equals the permutation character corresponding to the action of G on the set of left cosets of H in G given by $s(tH) = (st)H$ $(s, t \in G)$.

2. The induced character $(1_{\{e\}})^G$ equals the permutation character corresponding to the action of G on itself given by left multiplication. It is the character of the (left) regular representation of G. This character satisfies

$$(1_{\{e\}})^G(s) = \begin{cases} |G| & \text{if } s = e \\ 0 & \text{if } s \neq e. \end{cases}$$

3. As an illustration of Frobenius Reciprocity (Fact 11), we have $((1_{\{e\}})^G, \chi)_G = (1_{\{e\}}, \chi_{\{e\}})_{\{e\}} = \chi(e)$ for any irreducible character χ of G. Hence $(1_{\{e\}})^G = \sum_{\chi\in Irr(G)}\chi(e)\chi$ (cf. Fact 8 of subsection 4), that is, in the character of the regular representation (see Example 2), each irreducible character appears as a constituent with multiplicity equal to its degree.

6 REPRESENTATIONS OF THE SYMMETRIC GROUP

Definitions:

Given a natural number n, a tuple $\alpha = [\alpha_1, \ldots, \alpha_h]$ of nonnegative integers is a (**proper**) **partition** of n (written $\alpha \vdash n$) provided

- $\alpha_i \geq \alpha_{i+1}$ for all $1 \leq i < h$,
- $\sum_{i=1}^{h} \alpha_i = n$.

The **conjugate partition** of a partition $\alpha \vdash n$ is the partition $\alpha' \vdash n$ with ith component α'_i equal to the number of indices j for which $\alpha_j \geq i$. This partition is also called the **partition associated with** α.
Given two partitions $\alpha = [\alpha_1, \ldots, \alpha_h]$ and $\beta = [\beta_1, \ldots, \beta_k]$ of n, α **majorizes** (or **dominates**) β if

$$\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \beta_i$$

for each $1 \leq j \leq h$. This is expressed by writing $\alpha \succeq \beta$ (or $\beta \preceq \alpha$).
The **Young subgroup** of the symmetric group S_n corresponding to a partition $\alpha = [\alpha_1, \ldots, \alpha_h]$ of n is the internal direct product $S_\alpha = S_{A_1} \times \cdots \times S_{A_h}$, where S_{A_i} is the subgroup of S_n consisting of those permutations that fix every integer *not* in the set

$$A_i = \{1 \leq k \leq n \,|\, \sum_{j=1}^{i-1} \alpha_j < k \leq \sum_{j=1}^{i} \alpha_j\}$$

(an empty sum being interpreted as zero).
The **alternating character** of the symmetric group S_n is the character ϵ_n given by

$$\epsilon_n(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Let G be a subgroup of S_n and let χ be a character of G. The **generalized matrix function** $d_\chi : \mathbb{C}^{n\times n} \to \mathbb{C}$ is defined by

$$d_\chi(A) = \sum_{s\in G} \chi(s) \prod_{j=1}^{n} a_{js(j)}.$$

When $G = S_n$ and χ is irreducible, d_χ is called an **immanant**.

Facts:

The following facts can be found in [JK81, pp. 15, 35 - 37] or [Mer97, pp. 99 - 103, 214].

1. If χ is a character of the symmetric group S_n, then $\chi(\sigma)$ is an integer for each $\sigma \in S_n$.

2. *Irreducible character associated with a partition*: Given a partition α of n, there is a unique irreducible character χ_α that is a constituent of both the induced character $(1_{S_\alpha})^{S_n}$ and the induced character $((\epsilon_n)_{S_{\alpha'}})^{S_n}$. The map $\alpha \mapsto \chi_\alpha$ defines a bijection from the set of partitions of n to the set $Irr(S_n)$ of irreducible characters of S_n.

3. If α and β are partitions of n, then the irreducible character χ_α is a constituent of the induced character $(1_{S_\beta})^{S_n}$ if and only if α majorizes β.

4. If α is a partition of n, then $\chi_{\alpha'} = \epsilon_n \chi_\alpha$.

5. *Schur's inequality*: Let χ be an irreducible character of a subgroup G of S_n. For any positive semidefinite matrix $A \in \mathbb{C}^{n\times n}$, $d_\chi(A)/\chi(e) \geq det\, A$.

Examples:

1. $\alpha = [5, 3^2, 2, 1^3]$ (meaning $[5, 3, 3, 2, 1, 1, 1]$) is a partition of 16. Its conjugate is $\alpha' = [7, 4, 3, 1, 1]$.

2. $\chi_{[n]} = 1_{S_n}$ and $\chi_{[1^n]} = \epsilon_n$.

3. In the notation of Example 3 of subsection 4 we have $\chi_0 = \chi_{[4]}$, $\chi_1 = \chi_{[1^4]}$, $\chi_2 = \chi_{[2^2]}$, $\chi_3 = \chi_{[3,1]}$, and $\chi_4 = \chi_{[2,1^2]}$.

4. According to Fact 4, a partition α of n is self-conjugate (meaning $\alpha' = \alpha$) if and only if $\chi_\alpha(\sigma) = 0$ for every odd permutation $\sigma \in S_n$.

5. As an illustration of Fact 3, we have

$$(1_{S_{[2,1^2]}})^{S_4} = \chi_{[4]} + 2\chi_{[3,1]} + \chi_{[2^2]} + \chi_{[2,1^2]}.$$

The irreducible constituents of the induced character $(1_{S_{[2,1^2]}})^{S_4}$ are the terms on the right-hand side of the equation. Note that $[4]$, $[3, 1]$, $[2^2]$, and $[2, 1^2]$ are precisely the partitions of 4 that majorize $[2, 1^2]$ in accordance with the fact.

6. When $G = S_n$ and $\chi = \epsilon_n$ (the alternating character), $d_\chi(A)$ is the determinant of $A \in \mathbb{C}^{n\times n}$.

7. When $G = S_n$ and $\chi = 1_G$ (the principal character), $d_\chi(A) = \sum_{s\in G}\prod_{j=1}^n a_{js(j)}$ is called the **permanent** of $A \in \mathbb{C}^{n\times n}$, denoted *per* A.

8. The following open problem is known as the **Permanental Dominance** (or **Permanent-on-Top**) **Conjecture**: Let χ be an irreducible character of a subgroup G of S_n. For any positive semidefinite matrix $A \in \mathbb{C}^{n\times n}$, *per* $A \geq d_\chi(A)/\chi(e)$. (Cf. Fact 5 and Examples 6 and 7.)

References

[Gor94] D. Gorenstein. *The Classification of the Finite Simple Groups.* American Mathematical Society, Providence, 1994-.

[Isa94] I.M. Isaacs. *Character Theory of Finite Groups.* Academic Press, New York, 1976. Reprinted, Dover Publications, Inc., 1994.

[JK81] G. James and A. Kerber. *The Representation Theory of the Symmetric Group.* Encyclopedia of Mathematics and Its Applications **16**. Addison-Wesley Publishing Company, Reading, 1981.

[Mer97] R. Merris. *Multilinear Algebra.* Gordan and Breach Science Publishers, Amsterdam, 1997.

[Ser77] J.P. Serre. *Linear Representations of Finite Groups.* Springer-Verlag, New York, 1977.